

# Uncertainty, non-locality and Bell's inequality

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(February 1, 2008)

We derive a Bell-like inequality involving *all* correlations in local observables with uncertainty free states and show that the inequality *is violated* in quantum mechanics for EPR and GHZ states. If the uncertainties are allowed in local observables then the statistical predictions of hidden variable theory *is well respected* in quantum world. We argue that the uncertainties play a key role in understanding the non-locality issues in quantum world. Thus we can not rule out the possibility that a local, realistic hidden variable theory with statistical uncertainties in the observables might reproduce all the results of quantum theory.

Apart from certain conceptual difficulties and paradoxes quantum theory is the best working theory of nature that theorist have ever produced. Yet, some of the predictions of quantum mechanics are so counter intuitive that no common sense based idea fits into the realm of quantum world. It is the work of Einstein, Podolsky and Rosen (EPR) [1], that raised serious doubts about the completeness of the quantum theory. The heart of their argument was the locality and reality criterion that have lead to apparantly contradictory results within the quantum theory. Subsequent development was to build a more realistic and deterministic description of nature interms of hidden variables [2] which can reproduce all the results of quantum mechanics. Bell [3], however, proved that the reality criterion of observables (spin in case of singlet-states) put severe restrictions on the correlations of different observables in a local, hidden variable theory (LHV). Further, he showed that the quantum mechanical correlation do not obey these restrictions predicted by LHV theories. This is so called the violation of the Bell inequality in the quantum domain. From a sharpened principle of separability Clauser *et al* [4] have derived another inequality (CHSH) which is also violated in quantum mechanics. Recent experiment of Aspect *et al* [5] provides positive evidence in favour of quantum mechanics precluding the possible hidden variable theories. Therefore it is concluded that one can not in general reproduce the results of quantum theory with the help of any LHV theory. Yet, some believe that since experimental violation of Bell's inequality is not free from loopholes, it leaves some room for LHV theories [6].

In this paper, I look for a new inequality which involves correlation between observables of particle 1 and 2 as well as of observables of the particle 1 or 2. Bell's inequality and CHSH inequality involves correlations of distant observables and says that the quantum mechanical correlations do not obey the inequality that are satisfied by the correlations of LHV theories. Therefore the two descriptions of nature are incompatible with each other. But this precludes the answer to the question on the possible origin of the violation. *Is there any thing whose absence or presence in the linear combination of correlations will lead to violation or non-violation.*

In the past, variety of inequalities implied by local realism have been derived [7–10,12] which are, in general, violated by statistical predictions of quantum theory. It has been shown in [13] that all entangled pure states can violate Bell type inequality. Although the origin of conflict between quantum mechanical predictions and local realism is not very much clear, there is a hint that the non-local features of quantum world can be traced to complimentarity [14–16]. Chefles and Barnett [17] have shown that Cirel'son's inequality [14] can be derived using sum of Heisenberg uncertainty relation for appropriate quantum observables. Since complimentarity in physical quantities basically come from non-commutativity of quantum observables and the later is related to the uncertainties in observables, we conjecture that the presence of uncertainties in the observables could be the root cause of non-locality. Conversely, one can ask, if the uncertainties are introduced into any local realistic theory does the theory become non-local? It seems that this idea can be testably true.

We provide an inequality which involves both uncertainties and correlations of different observables in LHV theory and find that in quantum world this inequality is violated provided the states have no uncertainties in observables of hidden variable theories. When the states are allowed to have uncertainties in observables of HV theories then the inequality is not violated. Therefore it seems any measurement involving uncertainties and correlations can be modeled by LHV theories. If we talk of only correlations in hidden variable description then we cannot reproduce the results of quantum mechanics. Thus, our findings trace the *origin for the violation* of the classical Bell-like inequality, to the *absence of uncertainties*. This has been illustrated for the decay of two (EPR) as well as four spin-1/2 particle entangled states. (The later called Greenberger, Horn and Zeilinger (GHZ) state.) We have a feeling that the quantum mechanical correlations and uncertainties are intrinsically non-local nature in entangled states. Therefore, when we test an inequality of LHV theory involving both these uncertainties and correlations then the uncertainties allow the non-local nature of the theory to be displayed and hence the inequality is not violated in quantum mechanics. When uncertainties are absent then the resulting inequality in quantum domain shows predominantly local behaviour and hence it is violated.

In what follows we will derive a Bell-like inequality between different observable in local, deterministic, hidden variable theory and show how it is not respected by quantum mechanical predictions. In LHV theories all physical

quantities (observables) of a particle are entirely deterministic and denoted by  $O(\lambda)$ , where  $\lambda$  is the internal, hidden variable [18]. Since we have no access to the hidden parameters  $\lambda$ , only the average of  $O$  is of importance in real experiments. It is assumed that there exist a positive and normalised distribution  $\rho(\lambda)$ , such that the expectation value of the measurement of the observable  $O(\lambda)$ , is given by

$$O = \langle O(\lambda) \rangle = \int \rho(\lambda) O(\lambda) d\lambda. \quad (1)$$

Here,  $\langle O(\lambda) \rangle$  can be regarded as the average over the distributions of hidden variables. In our discussion we are not assuming the bivalued nature of the observables. Let us consider four local, real observables  $A(\lambda), B(\lambda), C(\lambda)$  and  $D(\lambda)$  of a composite system whose measurement will yield the averages as defined in (1). Now we introduce uncertainty in the measured value of any observable  $O(\lambda)$  in the following way

$$\Delta O^2 = \int \rho(\lambda) (O(\lambda) - \langle O(\lambda) \rangle)^2 d\lambda. \quad (2)$$

A point to be noted is that although  $O(\lambda)$ 's are deterministic observables, because of the unobservable nature of the hidden parameters we have uncertainties introduced in LHV theories. These uncertainties are *statistical in nature*, as the deviation in the measured value of an observable  $O$  is assumed to be due to a distribution in the values of the hidden variables over the ensemble of the system measured.

A state can be called dispersion free if for all  $\rho(\lambda)$  and for any observable  $O(\lambda)$  the relation  $\langle O(\lambda)^2 \rangle = \langle O(\lambda) \rangle^2$  is satisfied.

To derive an inequality involving both the uncertainties and correlations between the observables  $A(\lambda), B(\lambda), C(\lambda)$  and  $D(\lambda)$ , let us define two functions  $u(\lambda)$  and  $v(\lambda)$  as

$$\begin{aligned} u(\lambda) &= \left[ (A(\lambda) - B(\lambda)) - (A - B) \right] \rho(\lambda)^{\frac{1}{2}} \\ v(\lambda) &= \left[ (C(\lambda) + D(\lambda)) - (C + D) \right] \rho(\lambda)^{\frac{1}{2}}. \end{aligned} \quad (3)$$

By applying Schwartz inequality for the two function  $u(\lambda)$  and  $v(\lambda)$  we obtain

$$\left| \int \left[ (A(\lambda) - B(\lambda)) - (A - B) \right] \left[ (C(\lambda) + D(\lambda)) - (C + D) \right] \rho(\lambda) d\lambda \right|^2 \quad (4)$$

$$\leq \int \left[ (A(\lambda) - B(\lambda)) - (A - B) \right]^2 \rho(\lambda) d\lambda \cdot \int \left[ (C(\lambda) + D(\lambda)) - (C + D) \right]^2 \rho(\lambda) d\lambda. \quad (5)$$

The above inequality can be simplified and is given by

$$\begin{aligned} |E(A, C) + E(A, D) - E(B, C) - E(B, D)|^2 &\leq \\ &\left( \Delta A^2 + \Delta B^2 - 2E(A, B) \right) \left( \Delta C^2 + \Delta D^2 + 2E(C, D) \right) \end{aligned} \quad (6)$$

where  $E(A, C) = \int \rho(\lambda) A(\lambda) C(\lambda) d\lambda - \left( \int \rho(\lambda) A(\lambda) d\lambda \right) \cdot \left( \int \rho(\lambda) C(\lambda) d\lambda \right)$  is the correlation between the joint measurement of the objective realities of the observable  $A(\lambda)$  and  $C(\lambda)$ . In deriving (5) we have taken care of the locality of the distribution function and reality of the observables, together with uncertainties and correlations. Also, the positivity of the distribution function for all hidden parameters  $\lambda$  is crucial in arriving the above inequality. Since we have not assumed the dichotomy variables the above inequality applies for general observable of correlated systems.

If we assume that the states are dispersion free we put the uncertainties in local, objective realities to zero and the inequality in LHV theory would be given by

$$|E(A, C) + E(A, D) - E(B, C) - E(B, D)|^2 + 4E(A, B)E(C, D) \leq 0. \quad (7)$$

We will test the validity of the inequality (6) in quantum world, and see that the quantum mechanical predictions for the combination of correlations violate the above inequality.

Let us investigate the famous EPR-singlet state in the light of above inequality. The EPR-Bohm state represents the wavefunction of a disintegrated spin-0 quantum system into two spin- $\frac{1}{2}$  particles 1 and 2, is given by

$$|\Psi_{EPR}\rangle = \frac{1}{\sqrt{2}}(|+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2) \quad (8)$$

Here,  $|+\rangle_i$  and  $|-\rangle_i$ , ( $i = 1, 2$ ) are the two eigenstates of the Pauli matrix  $\sigma_z$  for  $i$ th particle. In this state if we measure the spin of particle 1 as up, then with certainty we know that the spin of 2 (which may be far away in space-like separated region) is down. Let us measure the spin of particle 1 in two directions  $\mathbf{a}$  and  $\mathbf{b}$  and spin of particle 2 along two directions  $\mathbf{c}$  and  $\mathbf{d}$ . Thus the quantum mechanical observables are  $A = \sigma_1 \cdot \mathbf{a}$ ,  $B = \sigma_1 \cdot \mathbf{b}$ ,  $C = \sigma_2 \cdot \mathbf{c}$  and  $D = \sigma_2 \cdot \mathbf{d}$ . In the Stern-Gelach analyser the single detection of the spins are denoted as  $E(A)$ ,  $E(B)$  for particle 1 and  $E(C)$ ,  $E(D)$  for particle 2, where  $E(A) = \langle \Psi_{EPR} | (\sigma_1 \cdot \mathbf{a}) | \Psi_{EPR} \rangle$  and others are similarly defined. They are all equal to zero. Further, the joint simultaneous detection of particles 1 and 2 (say spin of 1 along  $\mathbf{a}$  and spin of 2 along  $\mathbf{c}$ ) is denoted as  $E(A, C)$ . This is given by

$$E(A, C) = \langle \Psi_{EPR} | (\sigma_1 \cdot \mathbf{a})(\sigma_2 \cdot \mathbf{c}) | \Psi_{EPR} \rangle = -\mathbf{a} \cdot \mathbf{c} \quad (9)$$

and similarly for other observables we have  $E(A, D) = -\mathbf{a} \cdot \mathbf{d}$ ,  $E(B, C) = -\mathbf{b} \cdot \mathbf{c}$  and  $E(B, D) = -\mathbf{b} \cdot \mathbf{d}$ . The correlations between the observables  $A$  and  $B$  is

$$E(A, B) = \langle \Psi_{EPR} | (\sigma_1 \cdot \mathbf{a})(\sigma_1 \cdot \mathbf{b}) | \Psi_{EPR} \rangle = \mathbf{a} \cdot \mathbf{b} \quad (10)$$

and similarly for  $E(C, D) = \mathbf{c} \cdot \mathbf{d}$ . With these quantum mechanical correlations the inequality (6) takes the form

$$|\mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d} - \mathbf{b} \cdot \mathbf{c} - \mathbf{b} \cdot \mathbf{d}|^2 + 4(\mathbf{a} \cdot \mathbf{b})(\mathbf{c} \cdot \mathbf{d}) \leq 0 \quad (11)$$

The above inequality is clearly violated for certain angles between the unit vectors  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  and  $\mathbf{d}$ . For example if we let  $\mathbf{a} \cdot \mathbf{c} = \cos 30^\circ$ ,  $\mathbf{a} \cdot \mathbf{d} = \cos 120^\circ$ ,  $\mathbf{b} \cdot \mathbf{c} = \cos 140^\circ$ ,  $\mathbf{b} \cdot \mathbf{d} = \cos 160^\circ$ ,  $\mathbf{a} \cdot \mathbf{b} = \cos 120^\circ$  and  $\mathbf{c} \cdot \mathbf{d} = \cos 45^\circ$ , then we have on lhs a positive quantity. This also shows the violation of the Schwarz inequality. The violation of above inequality for EPR states, is an indication of the non-local nature of quantum mechanical correlations. Therefore, the above inequality can be regarded as a *Bell-like inequality* which has been obtained from Schwarz inequality by demanding dispersion free state of LHV theory.

To support our idea we can carry out the following test. If the *uncertainties are at the root of non-locality* then by allowing them in the Hidden variable theory we should get a *non-violation* in quantum world. Let us consider the inequality (5) with uncertainties and correlations together in the local, objective realities, then the resulting inequality for EPR state can be expressed as

$$|-\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{d}|^2 \leq 4(1 - \mathbf{a} \cdot \mathbf{b})(1 + \mathbf{c} \cdot \mathbf{d}), \quad (12)$$

where the uncertainty in the observable  $A$  in the EPR state is given by

$$\Delta A^2 = \langle \Psi | (\sigma_1 \cdot \mathbf{a})^2 | \Psi \rangle - \langle \Psi | (\sigma_1 \cdot \mathbf{a}) | \Psi \rangle^2 = 1 \quad (13)$$

and similarly for other observables it is also unity. For the same set of angles we can easily see that the above inequality is *not violated* for EPR states, because in this case we have on lhs the value .38 whereas on rhs it is 10.2.

Infact, this inequality is satisfied for all angles. This demonstrates the predictive power of local realistic theory with statistical uncertainties. Thus here is an inequality in LHV theory which respects the results of quantum theory. So what we are lead to? We can say that although correlations of quantum mechanical predictions are not reproduced by any LHV theories the uncertainties together with correlations when taken into account in LHV theories, it can mimic the predictions of quantum theory. The reason for this non-violation of the above inequality points to the existence of uncertainties in local observables, as conjectured earlier. Thus we have a transition from violation to the non-violation of an inequality in HV theory. Also, this paper pinpoints to the origin of the violation, namely the absence of uncertainties are responsible for the killing of Bell's non-local correlations in HV theory. Any HV theory, which try to reproduce the result of quantum theory must include the uncertainties and correlations between the observables.

Let us consider the example of the decay of four spin 1/2 particles as discussed by Greenberger, Horn and Zeilinger (GHZ) [19]. Imagine a composite system initially in the state  $m = 0$  in a magnetic field applied along z direction. The composite state then decays into two particles, one of them moving along +z and other along -z axis. Each of these particles carry spin 1. Further, each of these two particles undergoes decay into two spin 1/2 particles. The first set of particles (say 1 and 2) move along +z direction and second set (say 3 and 4) move along -z direction. GHZ have restricted the motion of these two set of particles along +z and -z direction so that there does not arise any spin-orbit coupling problem.

The entangled state for the decay of four spin 1/2 particles can be written as

$$|\Psi_{GHZ}\rangle = |1,0\rangle = \frac{1}{\sqrt{2}}(|++--\rangle - |--++\rangle). \quad (14)$$

Suppose we intend to measure the spin component of first set along some direction **a** and **b** and spin component of second set along **c** and **d**. For simplicity we assume that each of the unit vectors **a**, **b**, **c** and **d** lie in the x-y plane at angles  $\alpha, \beta, \gamma$  and  $\delta$ , respectively. Define the quantum mechanical observables corresponding to the objective realities  $A, B, C, D$  as follows.

$$A = \prod_{i=1}^2 (\sigma_i \cdot \mathbf{a}), \quad B = \prod_{i=1}^2 (\sigma_i \cdot \mathbf{b}), \quad C = \prod_{i=3}^4 (\sigma_i \cdot \mathbf{c}), \quad D = \prod_{i=3}^4 (\sigma_i \cdot \mathbf{d}) \quad (15)$$

The quantum mechanical correlations between different joint measurements of the observables  $A, C$  is defined as

$$E(A, C) = \langle \Psi_{GHZ} | \prod_{i=1}^2 (\sigma_i \cdot \mathbf{a}) \prod_{i=3}^4 (\sigma_i \cdot \mathbf{c}) | \Psi_{GHZ} \rangle, \quad (16)$$

and similarly between the others. Thus, in the state (13) these correlations are given by

$$E(A, C) = -\cos 2(\alpha - \gamma) \quad E(B, C) = -\cos 2(\beta - \gamma)$$

$$E(A, D) = -\cos 2(\alpha - \delta) \quad E(B, D) = -\cos 2(\beta - \delta)$$

and

$$E(A, B) = \cos 2(\alpha - \beta), \quad E(C, D) = \cos 2(\gamma - \delta) \quad (17)$$

To see the violation consider again the dispersion-free inequality as would be predicted by LHV theory, i.e. the inequality (6). For the problem under consideration this takes the form

$$|-\cos 2(\alpha - \gamma) - \cos 2(\beta - \gamma) + \cos 2(\alpha - \delta) + \cos 2(\beta - \delta)|^2 + 4 \cos 2(\alpha - \beta) \cos 2(\gamma - \delta) \leq 0 \quad (18)$$

We can check that for certain orientations of the unit vectors **a**, **b**, **c** and **d** this inequality is *violated*. For example, if we let the angles  $\alpha, \beta, \gamma$  and  $\delta$  be 45, 60, 120 and 150, respectively then (17) is violated. Therefore, the above inequality

can be regarded as a new Bell-like inequality for the decay of four spin 1/2 particles bringing out the non-local aspects of quantum correlations in four-particle entangled states.

Next we consider the inequality with uncertainties and correlation taken together. The inequality (5) for GHZ state can be expressed as

$$| -\cos 2(\alpha - \gamma) - \cos 2(\beta - \gamma) + \cos 2(\alpha - \delta) + \cos 2(\beta - \delta) |^2 \leq 4(1 - \cos 2(\alpha - \beta))(1 + \cos 2(\gamma - \delta)) \quad (19)$$

We can easily check that for some choice of orientations of the unit vectors **a**, **b**, **c** and **d** this inequality is *not violated*. For example, if we let the angles  $\alpha, \beta, \gamma$  and  $\delta$  be 45, 60, 120 and 150, respectively then lhs yields .0275 whereas rhs yields 6.804, showing the non-violation. Thus, our new inequality derived in HV theory not only respects the EPR state but also the state describing the decay of four spin 1/2 particles. Hence, an inequality involving both uncertainties and correlations can not discern the predictions of HV theory and quantum theory.

The example of EPR and GHZ states in the light of our new inequality says that uncertainties do play a fundamental role in HV theories as they do play in quantum theory. Any LHV description with statistical uncertainties can be in agreement with the predictions of quantum theory. Also, as I have shown here, that the absence of these uncertainties can actually lead to the violation of classical Bell-like inequality in quantum world.

In conclusion, we spell out that the Bell theorem stands only with respect to correlations of quantum mechanical origin. When we wish to construct a model of physical world involving quantum mechanical correlations only using a LHV theory, then it fails. However, if we wish to construct a model with statistical uncertainties and correlations in LHV theory, then it is possible in principle to do it. There is also a clue to this view from the work of Fine [20] (also see comments [21–23], where it has been shown that an equivalence exist between the statement that an inequality holds and the existence of a deterministic hidden variable theory. Therefore, we say that quantum world is essentially non-local in nature and the non-locality is rooted in the uncertainties of the quantum observables. And if the results of HV theories and quantum theory are in one to one correspondence then it would mean that the quantum mechanical uncertainties could be of statistical in origin.

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